Geometric Ray Tracing of a Paraxial Lens

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1. Introduction

The AESOP[1] computer algebra ray tracing package has been updated to include geometric tracing of a paraxial lens. This document describes ray direction and optical path changes upon traversal of the ideal optical element known as a paraxial lens. This is simply an infinitely thin, aberration-free lens in the paraxial approximation.

2. Ray Position and Direction

Figure 1 illustrates the geometry. The incident and refracted beam directions are $\vec{v}_i$ and $\vec{v}_r$. The horizontal position vectors in the lens plane and in the focal plane are $\vec{\rho} = \langle \xi, \eta \rangle$ and $\vec{r}(\vec{\psi}) = \langle x, y \rangle$, where $\vec{\psi} = \langle \psi_x, \psi_y \rangle$ is a rotation composition defined by Figure 2. The horizontal position vectors are illustrated in the lens plane in Figure 3. We also define $\vec{u} = \vec{r}(\vec{\psi}) - \vec{\rho}$.

In the paraxial approximation, we assume that ψ is small such that all input rays converge to a point on the focal plane defined by the intersection of the chief ray with the focal plane. An equivalent statement is that $\vec{r}(\vec{\psi})$ is independent of position $\vec{\rho}$ in the lens plane. This approximation will require an adjustment to the optical path, which will be derived in the next section.

From Figure 1 and Figure 2, we may write

$$\vec{r}(\vec{\psi}) = f \left\langle -\tan\psi_y,\ \tan\psi_x \right\rangle \tag{1}$$

where

$$\tan\psi_x = \frac{\vec{v}_i \cdot \hat{y}}{-\vec{v}_i \cdot \hat{z}}, \quad \tan\psi_y = \frac{\vec{v}_i \cdot \hat{x}}{\vec{v}_i \cdot \hat{z}} \tag{2}$$

For a skew ray in the paraxial approximation, the refracted ray direction vector can be decomposed as

$$\vec{v}_r = -\vec{\rho} + \left(-f\hat{z}\right) + \vec{r}(\vec{\psi}) \tag{3}$$

(This is seen most easily from Figure 1.) Let us now scale the distance vectors $\vec{\rho}$ and $\vec{r}(\vec{\psi})$ by the focal length *f*,

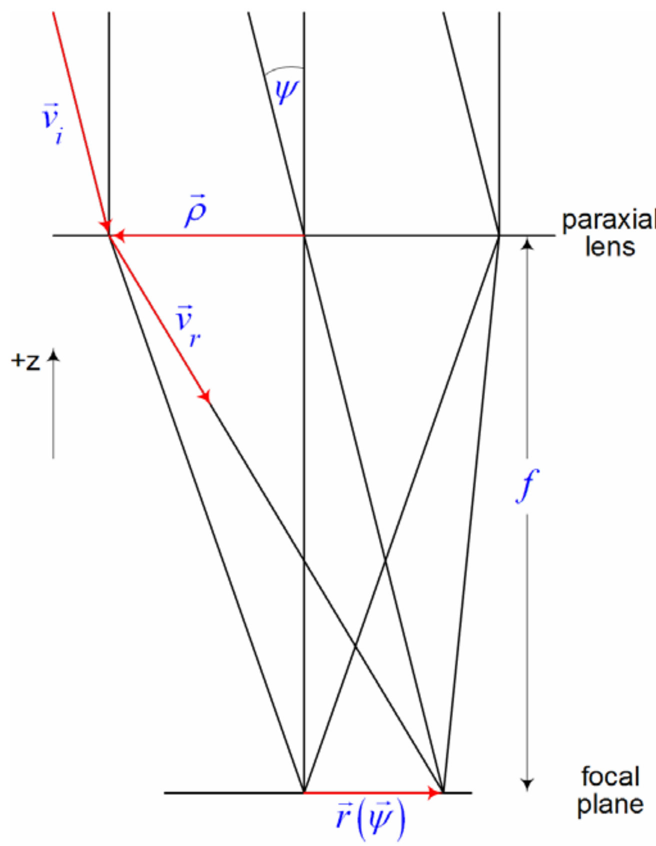

Figure 1

[1] http://arnold.usno.navy.mil/AESOP/

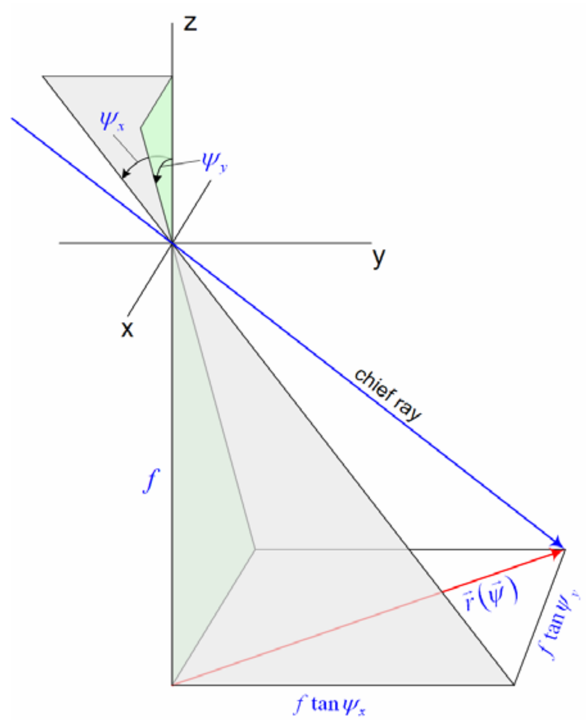

Figure 2

$$\frac{\vec{r}}{f} \to \vec{r}, \quad \frac{\vec{\rho}}{f} \to \vec{\rho} \tag{4}$$

Then we have the result

$$\vec{v}_r(\vec{\rho},\vec{\psi}) = -f \left\langle \rho\cos\varphi + \tan\psi_y,\ \rho\sin\varphi - \tan\psi_x, 1 \right\rangle \tag{5}$$

where $\tan\psi_x$ and $\tan\psi_y$ are given by eqs. (2).

3. Optical Path

For a perfect, paraxial lens, the optical path will be independent of the position $\vec{\rho}$ in the lens plane. Geometrically, the optical path of an axial ray from the lens to its focus is, assuming index $n=1$ in the medium,

$$\begin{aligned} s^2(0,\vec{\psi}) &= f^2 + \left|\vec{r}(\vec{\psi})\right|^2 \\ &= f^2\left(1+\tan^2\psi_x + \tan^2\psi_y\right) \\ &= f^2\left[1+\frac{(\vec{v}_i\cdot\hat{x})^2 + (\vec{v}_i\cdot\hat{y})^2}{(\vec{v}_i\cdot\hat{z})^2}\right] \end{aligned} \tag{6}$$

The geometric path for a skew ray is

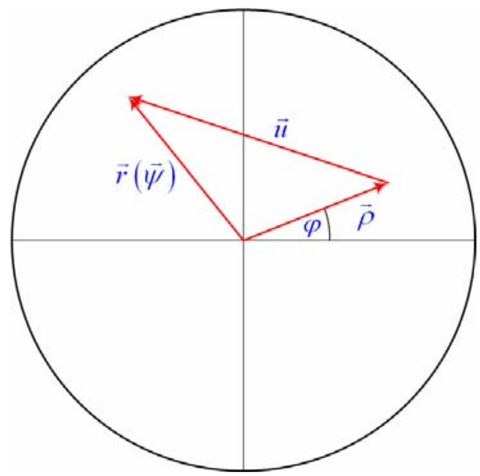

Figure 3

$$\begin{aligned} s^2(\vec{\rho},\vec{\psi}) &= f^2 + u^2 \\ &= f^2 + \rho^2 + r^2 - 2\vec{\rho}\cdot\vec{r} \end{aligned} \tag{7}$$

Thus, the optical path correction is

$$-\Delta s = s(\vec{\rho},\vec{\psi}) - s(0,\vec{\psi}) \tag{8}$$

In tracing a ray through a paraxial lens, Δs must be added to the calculated geometric path in order to produce a converging spherical wave front.

Normalizing again according to eq. (4), we have

$$\Delta s = f\left(\sqrt{1+r^2} - \sqrt{1+\rho^2+r^2-2\vec{\rho}\cdot\vec{r}}\right) \tag{9}$$

A Taylor expansion yields

$$\begin{aligned} \Delta s = f \Bigg\{ & \left(-\frac{\rho^2}{2} + \frac{\rho^4}{8} - \frac{\rho^6}{16} + \ldots\right) \\ & + \left[\left(\frac{\rho^2}{4} - \frac{3}{16}\rho^4\right) - \frac{3}{16}r^4\rho^2 + \ldots\right] \\ & + \left[1 - \frac{\rho^2}{2} + \frac{3}{8}\rho^4 - \left(\frac{1}{2} - \frac{3}{4}\rho^2\right)r^2 \right. \\ & \left. \quad + \frac{3}{8}r^4 + \ldots\right](\vec{r}\cdot\vec{\rho}) \\ & + \left(\frac{1}{2} - \frac{3}{4}\rho^2 - \frac{3}{4}r^2 + \ldots\right)(\vec{r}\cdot\vec{\rho})^2 \\ & + \frac{1}{2}(\vec{r}\cdot\vec{\rho})^3 + \ldots \Bigg\} \end{aligned} \tag{10}$$

From eqs. (1) and (2),

$$r^2 = \frac{\left(\vec{v}_i \cdot \hat{x}\right)^2 + \left(\vec{v}_i \cdot \hat{y}\right)^2}{\left(\vec{v}_i \cdot \hat{z}\right)^2} \tag{11}$$

Also, we have

$$\begin{aligned}\vec{r} \cdot \vec{\rho} &= -\xi \tan\psi_y + \eta \tan\psi_x \\ &= -\xi \frac{\vec{v}_i \cdot \hat{x}}{\vec{v}_i \cdot \hat{z}} - \eta \frac{\vec{v}_i \cdot \hat{y}}{\vec{v}_i \cdot \hat{z}} \\ &= -\frac{\vec{v}_i \cdot \vec{\rho}}{\vec{v}_i \cdot \hat{z}} \end{aligned} \tag{12}$$

Using eqs. (11) and (12), eqs. (9) and (10) may be expressed in terms of only the incident beam direction and the position in the lens plane, yielding the optical path correction $\Delta s\left(\vec{v}_i, \vec{\rho}\right)$. This is precisely what is needed in geometrical ray tracing, where ray position, direction, and optical path are calculated sequentially for each optical element in a system.

A special case exists for which it is not necessary to calculate for a skew ray either the intersection position or the optical path correction. This occurs if the surface following the paraxial lens is planar and parallel to the paraxial lens plane. Then we know that a) the intersection point will, for all skew rays, be identical to that of the chief ray, given by eqs. (1) and (2); and b) the optical path will be given simply by that of the chief ray, eq. (6). The circumstances of this special case, however, are rather uncommon for most systems, where the emphasis is on the effects of optical element misalignments. Thus, in general, the full skew ray calculations must be employed.